Making Sense with Diagrams: Students' Difficulties with Feature-Similar Problems

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Students experience a range of difficulties in generating effective diagrams. Hence, it is important to explore these difficulties so they can be addressed during instruction on diagram generation. A cross-study comparison of the results of two network tasks revealed that students experience similar difficulties on feature-similar but non-isomorphic tasks. Students' difficulties on these tasks appeared to be due to a lack of sense-making in mathematics rather than a difficulty with the problem structure or the generation of a particular type of diagram.

The use of the strategy *draw a diagram* is strongly advocated by mathematics educators as a tool for problem solving (Australian Education Council, 1991; National Council of Teachers of Mathematics [NCTM], 1998). A diagram is a particularly effective problem representation because it exploits spatial layout in a meaningful way, enabling complex processes and structures to be represented holistically (Winn, 1987). For some students, generating a diagram is the first step towards a successful solution (van Essen & Hamaker, 1990). However, students can also be misled by self-generated diagrams in the solution process (Antonietti & Angelini, 1991). Inadequate diagrammatic representations of problems may limit children's problem solving capabilities (Klahr, 1978), hence, it is important to investigate factors that influence problem representation (Goldman, 1986) and address these in an instructional program. Although there is a vast literature base on students' difficulties in some areas of mathematics (e.g., counting), the literature on primary students' difficulties in generating diagrams is scant. Hence, the purpose of this paper is to explore the various difficulties students' experience in the generation of diagrams.

Knowledge Acquisition through Diagrams

Representing written problem information on a diagram is initially a translation process that involves the decoding of linguistic information and the encoding of visual information. During this process, there is the potential for knowledge acquisition (Karmiloff-Smith, 1990) through the reorganisation of information (Weinstein & Mayer, 1986) and subsequent inference-making (Lindsay, 1995). For example, knowledge about family relationships that cannot be easily inferred from a description can be established on a family tree. Sternberg (1990) proposed that there are three knowledge acquisition components, namely, selective encoding, selective combination and selective comparison. Selective encoding relates to the relevance of the information that is represented. Some students' representations are unhelpful for problem solving because relevant problem information is not included (Dufoir-Janvier, Bednarz, & Belanger, 1987). For example, students may represent the surface (irrelevant) features of a problem rather than the structural (relevant) features (Dufoir-Janvier et al., 1987). Selective combination refers to how new information is integrated as a discrete entity. The diagram is an effective problem representation because problem information is represented by location on a plane, and hence, a large number of perceptual inferences about problem information is possible (Larkin & Simon, 1987). Selective comparison focuses on the relationship between new knowledge and prior knowledge. This component highlights the importance of background knowledge about general purpose diagrams that have applicability in problem solving (e.g., networks).

General Purpose Diagrams

General purpose diagrams assume an important role in mathematics because they provide representational frameworks that are applicable to a range of problem structures. These diagrams are networks, matrices, and hierarchies, and a range of diagrams that exhibit part-whole characteristics. Novick, Hurley, and Francis (1999) developed a theoretical framework for networks, matrices, and hierarchies, which described the conditions of applicability and distinguishing properties for these spatially-oriented diagrams. For example, a network is a path-like representation (e.g., a train line map), whereas a hierarchy is a tree-like representation (e.g., family tree). Part-whole diagrams were not included in Novick et al.'s framework because they have no unique external form. As effective instruction is informed by pedagogical content knowledge about students' difficulties and errors (Carpenter, Fennema, & Franke, 1996), it is important to explore students' difficulties and errors with each of the general purpose diagrams.

Design and Methods

This paper discusses the similarity between the results of structurally dissimilar problems from two separate studies.

Study 1 was an explanatory case study (Yin, 1994) in which the effect of instruction on children's use of diagrams in novel problem solving was investigated. The participants were 12 ten- to eleven-year-old Year 5 students. These students were presented with sets of five isomorphic novel problems during 30-minute interviews conducted before and after a series of lessons on the use of diagrams in problem solving. The interviews were video-taped and subsequently transcribed for analysis. One key finding of this study was the recognition of the variety of difficulties and errors students experienced in generating general purpose diagrams (Diezmann, 1999). As difficulties and errors in diagram generation constitute obstacles to students' problem solving performance, they were explored further in Study 2.

Study 2 focused on identifying the range of difficulties that students experience in generating general purpose diagrams for novel problems. This research was undertaken using an inductive theory-building framework, which requires description and explanation (Krathwohl, 1993). Data were collected until a "saturation" point was reached in which new observations did not provide further insight into the phenomenon. The participants in this study were a class of 25 ten- to eleven-year-old Year 5 students. Data comprised observations of individual children engaged in novel problem solving and the diagrams they generated during interview sessions. Over four videoed interview sessions, the students were presented with a total of eight tasks. Two tasks that could be represented using the same general purpose diagram (e.g., networks) were presented at each interview, which lasted approximately 20 minutes.

The findings of Study 1 and Study 2, respectively, revealed that difficulties in diagram generation can be due to (1) a lack of understanding of the problem structure, and (2) a lack of understanding of specific general purpose diagrams. During the analysis of the Study 2 data, it was noted that while there was limited correspondence between the difficulties students experienced on the two network tasks. However, there was an unanticipated similarity between students' difficulties on one of the pre-instruction network tasks in Study 1 and one of the network tasks in Study 2. As this correspondence can neither be explained by similarity in problem structures nor attributed solely to the general purpose diagram used, the findings from these network tasks were investigated further.

Results

The two network tasks that revealed similar diagram generation difficulties and errors are shown on Figure 1. Although the problems are not isomorphic, there is some obvious similarity in the features of the problems (e.g., height measurement and movement). Henceforth, these types of tasks are referred to as feature-similar tasks. The analysis of students' performance on these tasks revealed a range of diagram generation difficulties and errors. Three categories of difficulty emerged from students' performance on these tasks: (1) a lack of measurement sense, (2) a lack of spatial sense, and (3) a lack of number sense. Due to space limitations, the reporting of these difficulties is limited to one set of examples from each category.

Study 1 - The Koala: A sleepy koala wants to climb to the top of a gum tree that is 10 metres high. Each day the koala climbs up 5 metres, but each night, while asleep, slides back 4 metres. At this rate, how many days will it take the koala to reach the top?	**Study 2 - Bouncing Ball:** Sylvia dropped a tennis ball from a balcony 8 metres above the footpath. Each time the ball bounced it travelled half as high as on the previous bounce. Sylvia's brother caught the ball when it bounced exactly 1 metre from the footpath. How many times did the ball bounce?

Figure 1. The Koala and bouncing ball tasks.

A Lack of Measurement Sense

Although, measurement is integral to daily life and commonly encountered in applied contexts, some students' diagrams revealed the inadequacy of their measurement sense. One of the most pervasive errors on these feature-similar tasks was the incorrect measurement of "ground level". On both tasks, many children incorrectly labelled or referred to the ground height as one metre. This error is evident in Damien's diagram in which he incorrectly identified the first "mark" as one metre, instead of the second mark (see Figure 2). Damien's initial error was compounded because the ten marks on his tree only represent a nine metre tree rather than a ten metre tree as specified in The Koala task (see Figure 1). Jane's diagram also showed a lack of understanding of ground level. In contrast to Damien, Jane correctly indicated that ground height was zero metres. However her diagram suggests that she lacked an understanding of the location at which a ball rebounds. Her depiction of a bouncing ball shows the ball rebounding from one and two metres above the ground. Thus, Jane's representation is not consistent with the behaviour of a ball rebounding from ground level nor is it internally consistent because Jane has it rebounding from two different heights. In this task, Jane realised that if the ball was caught at one metre then ground level could not be one metre. However Jane may not have recognised her error if the ball had been caught at two metres. Both Damien and Jane's difficulties can be interpreted as a lack of measurement sense though either failure to identify ground level as zero metres, or the limited everyday knowledge that a ball rebounds from ground level and rebounds from the same height on each successive bounce. A lack of measurement sense was not restricted to difficulties with "ground level", further difficulties and errors were apparent in other students' diagrams.

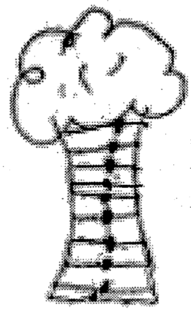
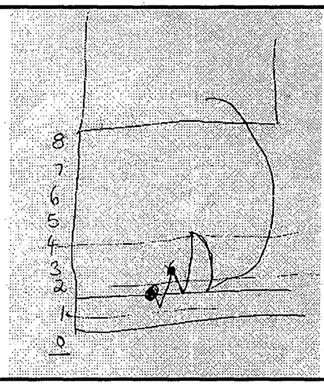
The Koala	The Bouncing Ball
	
Damien: He starts here. [Pointing to a **line at the base of the tree**] Researcher: He starts on that mark. What mark is that? What number? Damien: The first one. Researcher: The first what? Damien: The first part of the tree. Researcher: Okay. If you told me in metres how many metres would it be? Damien: **One metre.**	Jane: He caught it when it was ... [referring to the problem]...one metre. Researcher: One metre. So where is one metre? Jane: Away from...well, **there** is the ground [indicating the "1" on the diagram]. That's one metre away from the ground. No! **Zero's the ground..oh!**

Figure 2. A lack of measurement sense.

A Lack of Spatial Sense

Spatial sense is critical in the representation of information on diagrams (Battista, 1999). However students have a tendency to lack precision in their representation of position on a diagram. For example, although Helen understood the forward and backward motion of the koala, she did not keep track of the koala's precise location on the diagram (see Figure 3, Helen). Consequently, when Helen had completed the koala's movements she was unsure of its exact finishing position. Casey also had difficulty with spatial representation. While Casey accurately represented the pathway of the bouncing ball, she was unsure about what was represented by her lowest horizontal line (see Figure 3, Casey). Casey's uppermost line depicted the balcony and her series of shorter horizontal lines represented the metres from ground level to the balcony. Her inability to identify the lowest horizontal line as representing ground level is a particular concern given the difficulties that students experienced with "ground level" (see Figure 2). As diagrams capitalise on the use of graphics to structure and represent information spatially, the student who produces the diagram should understand each graphic component of a self-generated diagram. Thus, despite the differences in Helen's and Casey's difficulties, their diagrams both revealed a lack of spatial sense.

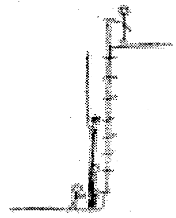
The Koala	The Bouncing Ball
	
Helen: He had to he climbed up another five [metres] and then he slept again and the second day when he climbed up —I mean the third day he climbed up to the top. Researcher: How do you know he climbed up to the top are you sure or could he have been just a bit lower? Helen: **He might have been a bit lower.**	Researcher: What's this line at the bottom? Casey: It's just **the finish of the thing**. It's just the —I don't know. I just —we just do it in class.

Figure 3. A lack of spatial sense.

A Lack of Number Sense

Students' diagrams also demonstrated a variety of numerically-based errors. The two errors that are shown in Figure 4 relate to confusion over concepts that are opposites. Kate's and Ellen 's errors respectively were in confusing "upper and lower", and "halving and doubling". While Kate recognised that 10 metres was a height constraint in The Koala problem, she failed to realise that 10 metres was the *upper* height constraint of the tree (the point the koala climbed to) rather than the *lower* height constraint (the point to which the koala slid back). This is considered to be a numerical error rather than a measurement error because Kate's measurements of the koala's movement were accurate. Ellen's error was in confusing halving with doubling. The Bouncing Ball problem states that on each bounce the ball travels "half as high as on the previous bounce" (see Figure 1), however Ellen doubled numbers commencing with "eight" when she should have been halving numbers. Ellen made two further numerical errors. First, she made a calculation error in doubling 64 to reach 127. Second, once Ellen had reached 127 she stopped doubling for no apparent reason. There is no correspondence between her answer of 127 and the finishing height of the ball of one metre. Both Kate's and Ellen's errors highlight the importance of sound number sense and the interrelationship among number, measurement, visualisation and language in problem representation.

In summary, the findings of this cross-study comparison revealed that a lack of understanding of specific mathematical features of problems results in similar difficulties being detected in the diagrams of feature-similar problems. Although each of the errors made by the students in the examples presented was to some extent unique, these difficulties are unified by a lack of sense-making about mathematics in measurement, space or number as the students attempted to portray particular problem information on a diagram. Given, the findings of cross-study regularities of difficulties and errors for network tasks, a search for difficulties and errors in feature-similar tasks for the other three general purpose diagrams will be followed up.

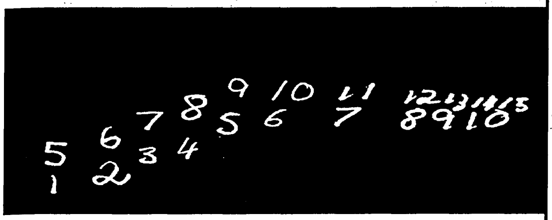
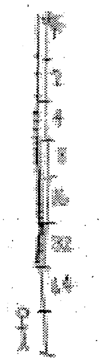
The Koala	The Bouncing Ball
	
Kate: And then he would climb up to number eight and down to four. And then he'd climb back up to number nine and slide back down to five and then he'd climb up to ten and slide back to six and then he'd climb up to **eleven.** I'd have to make the tree again. Eleven and slides back down to seven and then up to **twelve** and back to eight and then climb up there [13] and he'd slide back down to ...	Ellen: Well, each time I'm – I'm sort of bouncing it, **I'm trying to halve the amount** that it's going down at .... Ellen: [Draws closer and closer segments on the diagram and writes numbers 1, 2, 4, 8, 16, 32 and 64] 127 [answer] Researcher: So you're saying the ball bounced 127 times before he caught it. Ellen: Mmm. Researcher: Can you tell me how you worked that out? Ellen: Well, you have the one and then it bounces two times and then it bounces four times and you see **it's just sort of doubling** each time.

Figure 4. A lack of number sense.

Conclusion

Students' difficulties and errors in generating accurate and effective diagrams are generally associated with students' lack of expertise in diagrammatic representation (e.g., Dreyfus & Eisenberg, 1990). However the results of this cross-study comparison suggests that effective diagrammatic representation also depends on a sound mathematical knowledge base, which includes sense-making in mathematical situations.

The findings of this cross-study comparison have four key implications for problem solving instruction and future research on diagrams. First, knowing students' errors and difficulties in generating diagrams is an important component of effective instruction in diagram generation. Second, instruction should provide opportunities for the explication of graphic components of the diagram and the relationships depicted by them. Third, students' diagrams provide an insight into the strengths and weaknesses of their mathematical knowledge. Fourth, though diagrams can support the conceptualisation of a problem, they cannot substitute for a lack of basic mathematical knowledge. Thus, diagrams should be considered as both representations that stimulate reflection on the problem structure and reflections of students' mathematical knowledge.

The goal of mathematics education is to produce numerate citizens for the 21st century who have access to mathematics, who are able to reason analytically, and who can make informed decisions (NCTM, 1998). Hence, students' lack of sense-making in mathematics with representations, such as diagrams, is a particular cause for concern in an increasingly "data-drenched" and technological society (Steen, 1997):

> As information becomes even more quantitative and as society relies increasingly on computers and the data they produce, an innumerate citizen of today is as vulnerable as the illiterate peasant in Gutenberg's time. (p. xv)

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